27.4.20

I. Basic notions

Recall:

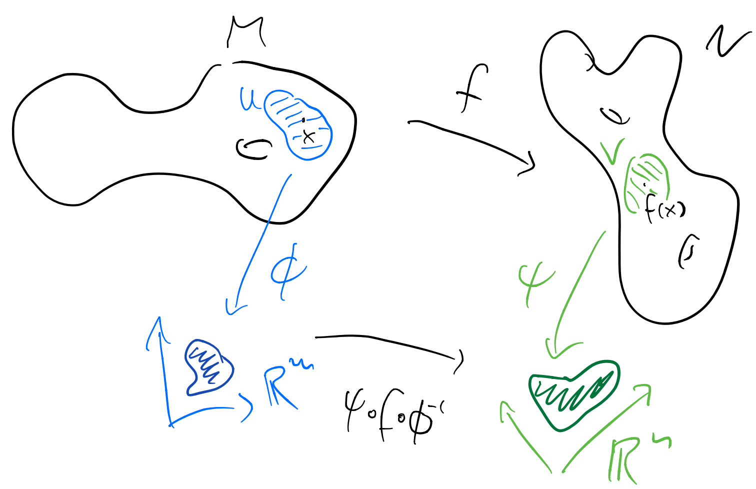
- smooth manifold := topological space M covered by charts s.t. transition maps are smooth.

- $f: M \to N$ is smooth at $x \in M$

  $\iff$ for all charts ϕ around x and ψ around $f(x) \in N$ we have that

  $$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \supset \phi(U) \to \psi(V) \subset \mathbb{R}^n$$

  is smooth at $\phi(x) \in \mathbb{R}^m$.



- T_xM, the tangent space at $x \in M$

$$:= \begin{cases} \{\gamma : (-\varepsilon, \varepsilon) \to M \mid \gamma(0) = x\} /_{\sim} \quad \sim \;\Leftrightarrow\; \text{"}\dot\gamma(0) = \dot{\tilde\gamma}(0)\text{"} \\ \text{or} \\ \{D : C^\infty(M, \mathbb{R}) \to \mathbb{R} \mid D(fg) = D(f)\cdot g(x) + f(x)\cdot D(g) \end{cases}$$

- TM, the tangent bundle of M

$$:= \bigsqcup_{x \in M} T_x M = \{ (x,v) \mid v \in T_x M \} \xrightarrow{\pi} M, \quad \pi(x,v) = x$$

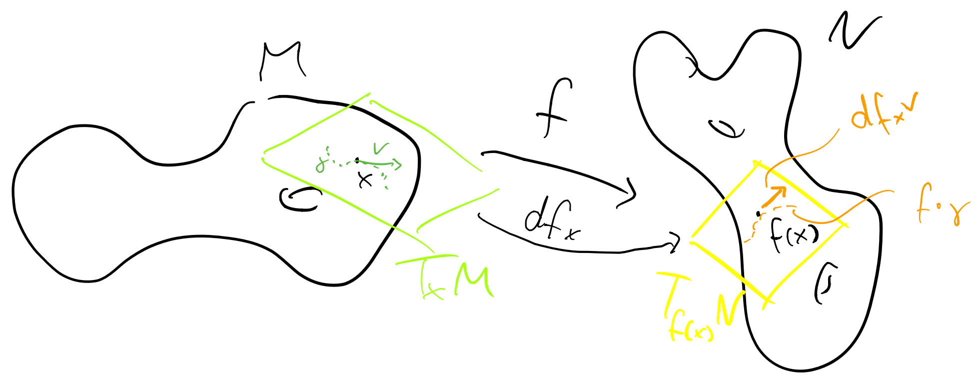
(is a vector bundle over M)

- The differential of a smooth map $f: M \to N$ is the smooth map

$$df : TM \to TN$$

s.t. $\forall x \in M$:

- $df(T_x M) \subset T_{f(x)} N$
- $df_x := df|_{T_x M} : T_x M \to T_{f(x)} N$ is linear



1. Def.: $x \in M$ is a critical point of $f : M \to N$ if

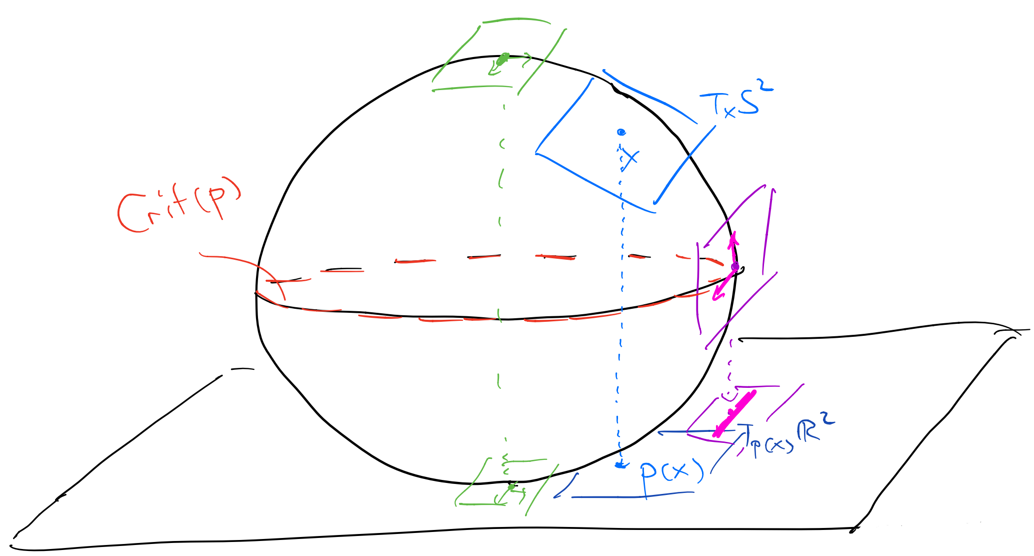
$$\operatorname{rank}(df_x) < \min\{\dim M, \dim N\},$$

otherwise x is a regular point. Correspondingly, $f(x)$ is called a critical or regular value of f.

We call $k := \min\{\dim M, \dim N\} - \operatorname{rank}(df_x)$ the corank of f at x.

e.g.

- examples from last lecture, $f(x) = x^2$, $g(x) = x^3$ at 0.
  (What about the "double point"?)
- $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$: $\operatorname{Crit}(f) = \{0\}$
- $p : S^2 \to \mathbb{R}^2$ vertical projection
  $\{\text{critical points}\} = \{\text{equator}\}$



How to compare / classify smooth maps?

2. Def.: Let $f_i : M_i \to N_i$ be smooth, $i = 1, 2$.

Then f_1 and f_2 are topologically equivalent if there are homeomorphisms $r : M_1 \to M_2$, $\ell : N_1 \to N_2$ s.t.

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "r"'] \arrow[dr, phantom, "\circlearrowright"] & N_1 \arrow[d, "\ell"] \\ M_2 \arrow[r, "f_2"] & N_2 \end{tikzcd}$$

, i.e. $f_2 = \ell \circ f_1 \circ r^{-1}$

e.g.: $f_i : \mathbb{R} \to \mathbb{R}$ $\quad f_1(x) = x^2$, $f_2(x) = x^4$

are topol. equivalent

$$\ell = id_{\mathbb{R}}, \quad r^{-1}(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "x^2"] \arrow[d, "r"'] & \mathbb{R} \arrow[d, "\ell"] \\ \mathbb{R} \arrow[r, "x^4"] & \mathbb{R} \end{tikzcd}$$

but the singularity of f_2 at 0

is degenerate / unstable!

$f_2''(0) = 0$

consider a perturbation of f_2, for instance:

$g_\varepsilon(x) = x^4 + \varepsilon x^2$ for $\varepsilon \ll 0$

$$\mathrm{Crit}(g_\varepsilon) = \{ 4x^3 + 2\varepsilon x = 0 \}$$

$$= \{0\} \cup \{ x^2 + \tfrac{\varepsilon}{2} = 0 \}$$

__3. Def:__ A smooth equivalence between $f_i : M_i \to N_i$, $i = 1,2$, is a commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "r"'] \arrow[dr, phantom, "\circlearrowleft"] & N_1 \arrow[d, "\ell"] \\ M_2 \arrow[r, "f_2"] & N_2 \end{tikzcd}$$

with r and ℓ diffeomorphisms.

Put abstractly, the group $\mathrm{Diff}(M) \times \mathrm{Diff}(N)$ acts on on $C^\infty(M,N)$ by the left-right action

$$(r,\ell).f = \ell \circ f \circ r^{-1}$$

and f_1, f_2 are smoothly equivalent if they belong to the same orbit of the left-right action.

"__4. Def:__" f is (smoothly/ left-right) stable if every map "close" to it is smoothly equivalent to it.

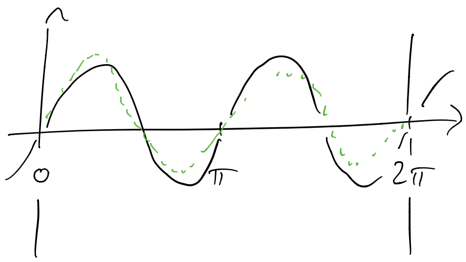
$\Leftrightarrow$ f stable if its left-right orbit is "open".

__Remark:__ Topological stability is defined similarly (replace diffeom. by homeom. everywhere). $\{\ell\text{-}r \text{ stable}\} \subsetneq \{\text{top. stable}\}$.

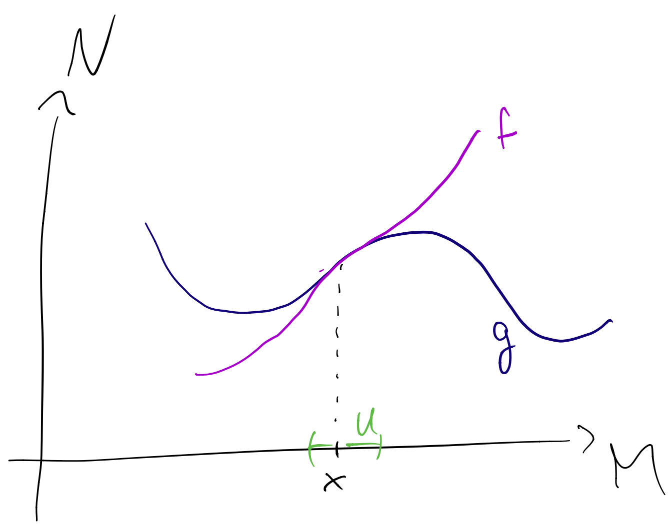
e.g.:

- $p: S^2 \to \mathbb{R}^2$ is left-right (and topologically) stable
- $q: S^1 \to \mathbb{R}$, $q(x) = \sin 2x$, is both left-right and topol. unstable.

$S^1 = \mathbb{R}/x \sim x+2\pi$



It is often convenient (or necessary) to work with a local version of stability. This leads to the notion of map-germs / germs of maps:



f and g are the "same" around x.

5. Def.: Let $f, g : M \to N$ and $x \in M$. We say f and g have the same germ at x if $\exists\, U$ neighborhood of x s.t. $f|_U = g|_U$. This defines an equivalence relation on $C^\infty(M,N)$ and its equivalence classes are called (smooth) germs of maps or map-germs at x.

6. Def.: • A map-germ f_1 at x_1 is (left-right/ smoothly) equivalent to a map-germ f_2 at x_2 if there exist diffeomorphism-germs r at x_1, sending x_1 to x_2, and l at $f(x_1)$, sending $f(x_1)$ to $f(x_2)$, such that $\exists\, U_1^*$ a neighbourhood of x_2, with

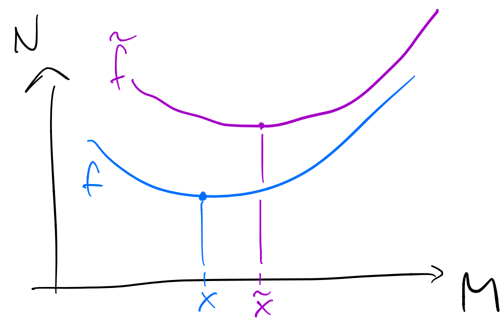
$$l \circ f_1 \circ r^{-1} = f_2 \text{ on } U.$$

The equivalence class of a germ at a critical point is called a singularity.

⌈$*$ depending on the germs of f_1, f_2 and r, l.⌋

- A map-germ $f: M \to N$ at $x \in M$ is (left-right/smoothly) stable if $\exists$ neighbourhoods U of x and "E of f in $C^\infty(M,N)$" such that

$\forall \tilde{f} \in E \quad \exists \tilde{x} \in U$: the germ of $\tilde{f}$ at $\tilde{x}$ is equivalent to the germ of f at x.



Remark: Topological equivalence and stability of germs are defined similarly...

e.g. The map germs $f_1(x) = x^2$ and $f_2(x) = x^4$ at 0 are topologically equivalent, but not left-right equivalent.
The germ f_1 at 0 is stable while the germ f_2 at 0 is not.